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Warping NMPC for Online Generation and Tracking of Optimal Trajectories ⋆

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Abstract: Generation of feasible and optimal reference trajectories is crucial in tracking Nonlinear Model Predictive Control. Especially, for stability and optimality in presence of a time varying parameter, adaptation of the tracking trajectory has to be implemented. General approaches are real-time generation of trajectories or switching between a discrete set of precomputed trajectories. In order to circumvent the operational efforts of these methods for a special type of dynamical systems, we propose time warping as an alternative approach. This algorithm implements online generation of tracking trajectories by warping a single precomputed reference. In detail, warpable systems, feasibility and optimality of trajectories and the controller implementation are discussed. Finally, as an application example, simulation results of a tethered kite system for airborne wind energy generation are presented.

Keywords: Predictive control, optimal trajectory, renewable energy systems, optimization.

1. INTRODUCTION

In the last decade, with the rise of computational power, tracking *Nonlinear Model Predictive Control (NMPC)* (Rawlings and Mayne (2009)) has been shown to be a viable and efficient solution for multivariable control of different nonlinear systems. In its standard implementation, the algorithm comprises two steps: first, offline generation of optimal or feasible trajectories, and then, online tracking of these trajectories by means of NMPC. While the concept has been successfully implemented and demonstrated in different scenarios (Guerreiro et al. (2009)), it suffers from robustness and stability issues. In particular, as the trajectories are computed offline, the controller lacks online adaptation to real disturbances and model mismatches. Moreover, in contrast with *economic NMPC* and despite offering a more stable controller than the latter, it is not able to guarantee optimality of the tracked trajectories. In order to potentially improve the performance of tracking NMPC in several different areas, generating optimal trajectories in real time is highly desired though demanding and still subject to current research activities (Hehn and D'Andrea (2011)).

In this paper, we regard a tracking NMPC scheme controlling a dynamical system whose *equations of motion (EOM)* depend on a time variable parameter $p(t) \in \mathbb{R}$. General approaches to include the $p(t)$ dependence are based either on real time generation of optimal trajectories or switching between a discrete set of precomputed trajectories for different p values (Ilzhoefer et al. (2007)). However, as both approaches suffer from operational drawbacks, we propose an algorithm to perform online generation of feasible and optimal tracking trajectories. By exploiting the specific structure of the considered dynamical systems, tracking trajectories for any $p(t)$ value are obtained by time warping a single reference trajectory y_{ref} computed for the reference value p_{ref}.

In order to derive and illustrate the algorithm, this paper is organized as follows: In a first section, we introduce the warping operation and define the class of warpable systems and their particular property of preserving feasibility under the warping operation. The second section deals with the class of *warpable optimal control problems (WOCP)* as a type of optimization problems to generate optimal trajectories for warpable systems; in detail optimality and limitations due to constraints are discussed. In the third section, the implementation of warping NMPC based on the previously introduced theory is explained. Finally, the last section illustrates the performance of the algorithm applied to an airborne wind energy system that is subject to a varying wind speed v_{w}.

For notational simplicity, concatenations of several vectors, e.g. $[x^\top, y^\top]^\top$, will be shortened as (x, y).

2. WARPABLE SYSTEMS

Definition 1. (Warped Time Frame τ). Consider a real time frame t used to describe the motion of a dynamical system. A *warped time frame* τ with respect to t is defined by the relation between the time velocities $\mathrm{d}t$ and $\mathrm{d}\tau$. This relation is called warping factor $\dot{w}(t)$ and is defined as:

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$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \dot{w}(t), \tag{1}$$

with $\dot{w}(t) > 0,\ \mathrm{d}t > 0$ and $\mathrm{d}\tau > 0$.

Remark 2. Time transformations from t to τ can be computed by $\tau = w(t) = \int_0^t \dot{w}(t')\mathrm{d}t'$. Likewise, τ can be warped back to obtain t by using $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\dot{w}(t)}$, i.e. the warping operation is bidirectional.

Regard a general dynamical system defined by the EOM $\dot{x}(t) = \Phi(x(t), u(t), p(t), t)$, with t representing the time, $x \in \mathbb{R}^{n_x}$ the system state, $u \in \mathbb{R}^{n_u}$ the system input, and $p \in \mathbb{R}^{n_p}$ the time dependent parameters.

Definition 3. (Warpable Dynamical System). We define the system to be a *warpable dynamic system* if the EOM can be expressed as:

$$\begin{aligned}\dot{x}(t) &= p(t)\, f\big(x(t), u_1(t)\big) + l\big(x(t), u_1(t)\big)\, u_2(t) \\ &= \quad p(t)\, g(t) \quad + \quad s(t)\, u_2(t),\end{aligned} \tag{2}$$

$$\text{with} : p(t) \in \mathbb{R}, \quad u(t) = \big(u_1(t), u_2(t)\big) \in \mathbb{R}^{n_{u_1}+n_{u_2}},$$
$$f : \mathbb{R}^{n_x+n_{u_1}} \longrightarrow \mathbb{R}^{n_x}, \quad l : \mathbb{R}^{n_x+n_{u_1}} \longrightarrow \mathbb{R}^{n_x \times n_{u_2}}.$$

Lemma 4. (Time Warped Dynamical System). Given the solution $\big(x_{\mathrm{ref}}(\tau), u_{\mathrm{ref1}}(\tau), u_{\mathrm{ref2}}(\tau)\big)$ for a reference system:

$$\begin{aligned}\dot{x}_{\mathrm{ref}}(\tau) &= p_{\mathrm{ref}}\, f\big(x_{\mathrm{ref}}(\tau), u_{\mathrm{ref1}}(\tau)\big) \\ &\quad + l\big(x_{\mathrm{ref}}(\tau), u_{\mathrm{ref1}}(\tau)\big)\, u_{\mathrm{ref2}}(\tau) \\ &= p_{\mathrm{ref}}\, g_{\mathrm{ref}}(\tau) + s_{\mathrm{ref}}(\tau)\, u_{\mathrm{ref2}}(\tau),\end{aligned} \tag{3}$$

a solution for the general system (2) is given by:

$$x(t) = x_{\mathrm{ref}}\big(w(t)\big), \tag{4a}$$
$$u_1(t) = u_{\mathrm{ref1}}\big(w(t)\big), \tag{4b}$$
$$u_2(t) = \dot{w}(t)\, u_{\mathrm{ref2}}\big(w(t)\big), \tag{4c}$$

where the warping factor between t and τ is defined by:

$$\dot{w}(t) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{p(t)}{p_{\mathrm{ref}}} \quad \text{and} \quad w(t) = \int_0^t \frac{p(t')}{p_{\mathrm{ref}}}\mathrm{d}t' = \tau. \tag{5}$$

Note: without loss of generality, the initial condition $x(0) = x_{\mathrm{ref}}(0)$ is assumed.

Proof.

$$\begin{aligned}\dot{x}(t) &\overset{(4a)}{=} \frac{\mathrm{d}x_{\mathrm{ref}}}{\mathrm{d}\tau}\Big|_{\tau=w(t)} \dot{w}(t) \\ &\overset{(3)}{=} \dot{w}(t)\, p_{\mathrm{ref}}\, f\Big(x_{\mathrm{ref}}\big(w(t)\big), u_{\mathrm{ref1}}\big(w(t)\big)\Big) \\ &\quad + l\Big(x_{\mathrm{ref}}\big(w(t)\big), u_{\mathrm{ref1}}\big(w(t)\big)\Big)\, \dot{w}(t)\, u_{\mathrm{ref2}}\big(w(t)\big) \\ &\underset{(5)}{\overset{(4a\text{-}4c)}{=}} p(t)\, f\big(x(t), u_1(t)\big) + l\big(x(t), u_1(t)\big)\, u_2(t).\end{aligned} \tag{6}$$

It is important to note that, in the defined system, the p parameter determines the speed of the system's time evolution. To have a first glance of this time warping interpretation, Fig. 4 and Fig. 5 depict a time series of states and controls, respectively.

3. OPTIMALITY OF WARPED TRAJECTORIES

Regard a general *Optimal Control Problem (OCP)* defined in a time frame t:

$$\underset{y(\cdot)}{\text{minimize}} \quad J\big(y(t)\big) = \int_0^T L\big(x(t), u(t), p(t)\big)\,\mathrm{d}t \tag{7a}$$
$$\text{subject to} \quad \Phi\big(x(t), u(t), p(t)\big) = \dot{x}(t), \quad t \in [0, T], \tag{7b}$$
$$h\big(x(t), u(t)\big) \leq 0, \qquad t \in [0, T], \tag{7c}$$
$$r\big(x(0), x(T)\big) \leq 0, \tag{7d}$$

with

$$y(t) = \big(x(t), u(t)\big). \tag{7e}$$

Definition 5. (Warpable Optimal Control Problem). If it holds that:

(i) The dynamical system of the OCP is warpable:
$$\Phi(\cdot) = p(t)\, f\big(x(t), u_1(t)\big) + l\big(x(t), u_1(t)\big)\, u_2(t). \tag{8a}$$
(ii) $p(t)$ is constant in the time interval $[0, T]$.
(iii) The OCP path constraints are independent of $u_2(t)$:
$$h\big(x(t), u(t)\big) = h\big(x(t), u_1(t)\big). \tag{8b}$$
(iv) The cost of the OCP can be written as:
$$J\big(y(t)\big) = \int_0^T L_1(p)\, L_2\left(x(t), u_1(t), \frac{u_2(t)}{p}\right)\mathrm{d}t. \tag{8c}$$

Then, we define the OCP to be a *Warpable Optimal Control Problem (WOCP).*

Remark 6. If (7) is a WOCP, it can be expressed as:

WOCP(p):

$$\underset{y(\cdot)}{\text{minimize}} \quad \int_0^T L_1(p)\, L_2\left(x(t), u_1(t), \frac{u_2(t)}{p}\right)\mathrm{d}t \tag{9a}$$
$$\text{subject to} \quad p\, g(t) + s(t)\, u_2(t) = \dot{x}(t), \quad t \in [0, T], \tag{9b}$$
$$h\big(x(t), u_1(t)\big) \leq 0, \qquad t \in [0, T], \tag{9c}$$
$$r\big(x(0), x(T)\big) \leq 0. \tag{9d}$$

Theorem 7. (Optimality of Warpable Dynamical Systems). Regard the WOCP in a reference time frame:

$$\begin{aligned}\underset{y_{\mathrm{ref}}(\cdot)}{\min} \quad & \int_0^{\bar{\tau}} L_1(p_{\mathrm{ref}})\, L_2\left(x_{\mathrm{ref}}(\tau), u_{\mathrm{ref1}}(\tau), \frac{u_{\mathrm{ref2}}(\tau)}{p_{\mathrm{ref}}}\right)\mathrm{d}\tau \\ \text{s.t.} \quad & p_{\mathrm{ref}}\, g_{\mathrm{ref}}(\tau) + s_{\mathrm{ref}}(\tau)\, u_{\mathrm{ref2}}(\tau) = \dot{x}_{\mathrm{ref}}(\tau), \quad \tau \in [0, \bar{\tau}], \\ & h\big(x_{\mathrm{ref}}(\tau), u_{\mathrm{ref1}}(\tau)\big) \leq 0, \qquad \tau \in [0, \bar{\tau}], \\ & r\big(x_{\mathrm{ref}}(0), x_{\mathrm{ref}}(\bar{\tau})\big) \leq 0.\end{aligned} \tag{10}$$

Given the optimal solution of the reference problem:

$$y^*_{\mathrm{ref}}(\tau) = \big(x^*_{\mathrm{ref}}(\tau), u^*_{\mathrm{ref1}}(\tau), u^*_{\mathrm{ref2}}(\tau)\big), \tag{11}$$

then, the warped trajectory of $y^*_{\mathrm{ref}}(\tau)$:

$$y_p(t) = \big(x_p(t), u_{p1}(t), u_{p2}(t)\big), \tag{12}$$

with constant warping factor:

$$\dot{w}(t) = \frac{p}{p_{\mathrm{ref}}} = \dot{w}, \tag{13}$$

and with warping transformations defined by (4a–4c), is the optimal solution of (9), i.e.:

$$x_p(t) = \ x^*_{\mathrm{ref}}\big(w(t)\big) \ = x^*(t), \tag{14a}$$
$$u_{p1}(t) = \ u^*_{\mathrm{ref1}}\big(w(t)\big) \ = u^*_1(t), \tag{14b}$$
$$u_{p2}(t) = u^*_{\mathrm{ref2}}\big(w(t)\big)\dot{w} = u^*_2(t). \tag{14c}$$

Note that, since the warping factor is time independent, time warping becomes a linear transformation:

$$\tau = \int_0^t \frac{p}{p_{\mathrm{ref}}}\,\mathrm{d}t' = \frac{p}{p_{\mathrm{ref}}}t \implies \bar{\tau} = w(T) = \frac{p}{p_{\mathrm{ref}}}T. \tag{15}$$

Proof.

$$J\big(y_{\text{ref}}(\tau)\big) = \int_0^{\bar{\tau}=w(T)} L_1(p_{\text{ref}})\, L_2\left(x_{\text{ref}}(\tau), u_{\text{ref1}}(\tau), \frac{u_{\text{ref2}}(\tau)}{p_{\text{ref}}}\right) \mathrm{d}\tau$$

$$= \int_0^T L_1(p_{\text{ref}}) L_2\left(x_{\text{ref}}\big(w(t)\big), u_{\text{ref1}}\big(w(t)\big), \frac{u_{\text{ref2}}\big(w(t)\big)}{p_{\text{ref}}}\right) \dot{w}\, \mathrm{d}t$$

$$\overset{\substack{(4c)\\(13)\\(4a)\\(4b)}}{=} \frac{p\, L_1(p_{\text{ref}})}{p_{\text{ref}}\, L_1(p)} \int_0^T L_1(p)\, L_2\left(x(t), u_1(t), \frac{u_2(t)}{p}\right) \mathrm{d}t$$

$$= \frac{p\, L_1(p_{\text{ref}})}{p_{\text{ref}}\, L_1(p)}\, J\big(y(t)\big). \tag{16}$$

In other words, the cost function of a general WOCP(p) and the cost function of the reference WOCP(p_{ref}) counterpart just differ in a constant factor. Therefore, if the problems were unconstrained, the optimal solutions of the WOCP(p_{ref}) and the WOCP(p) are related by warping.

Nevertheless, the WOCP is a constrained minimization problem; as a result, to prove that both WOCPs share the same solution, the constraints have to also be equivalent.

Dynamic constraints:

$$\begin{aligned} \dot{x}_{\text{ref}}(\tau) - p_{\text{ref}}\, g_{\text{ref}}(\tau) + s_{\text{ref}}(\tau)\, u_{\text{ref2}}(\tau) &= 0, \\ \Longleftrightarrow \dot{x}(t) \quad - p\, g(t) \qquad + s(t)\, u_2(t) \qquad &= 0. \end{aligned} \tag{17}$$

holds directly by Theorem 4.

Path and boundary constraints:

$$h\big(x_{\text{ref}}(\tau), u_{\text{ref1}}(\tau)\big) \overset{(4a,\,4b)}{\underset{(15)}{=}} h\big(x(t), u_1(t)\big) \le 0, \tag{18}$$

$$r\big(x_{\text{ref}}(0), x_{\text{ref}}(\bar{\tau})\big) \overset{(4a)}{\underset{(15)}{=}} r\big(x(0), x(T)\big) \le 0. \tag{19}$$

Feasible time set:

$$\tau \in [0, \bar{\tau}] \overset{(15)}{\Longrightarrow} \frac{p}{p_{\text{ref}}} t \in [0, \frac{p}{p_{\text{ref}}} T] \Longrightarrow t \in [0, T]. \tag{20}$$

As it can be seen from (19–20), the constraints in both WOCPs represent the exact same information. Therefore, considering (16–20), solving the reference WOCP given by (10) is equivalent to solving the real WOCP given by (9). As a result, the optimal solutions of both WOCPs, $y^*_{\text{ref}}\big(w(t)\big)$ and $y^*(t)$, are equivalent and related by (14).

Definition 8. (Semi-WOCP (SWOCP)). Consider a general WOCP as given by (9a-9d). The problem extension of adding $u_2(t)$-dependent path constraints as $h_2(\cdot)$ is defined as *Semi-Warpable Optimal Control Problem (SWOCP)* and can be expressed as:

SWOCP(p):

$$\min_{y(\cdot)} \quad \int_0^T L_1(p)\, L_2\left(x(t), u_1(t), \frac{u_2(t)}{p}\right) \mathrm{d}t \tag{21a}$$

$$\text{s.t.} \quad p\, g(t) + s(t)\, u_2(t) = \dot{x}(t), \quad t \in [0, T], \tag{21b}$$

$$h\big(x(t), u_1(t)\big) \le 0, \quad t \in [0, T], \tag{21c}$$

$$h_2\big(x(t), u_1(t), u_2(t)\big) \le 0 \quad t \in [0, T], \tag{21d}$$

$$r\big(x(0), x(T)\big) \le 0. \tag{21e}$$

It is important to note that, by adding u_2-dependent constraints, a warped version y_p of an optimal reference trajectory $y^*_{\text{ref}}(\tau)$ does not necessarily satisfy feasibility:

$$\begin{aligned} & h_2\big(x^*_{\text{ref}}(\tau), u^*_{\text{ref1}}(\tau), u^*_{\text{ref2}}(\tau)\big) \le 0 \\ \Longrightarrow\ & h_2\big(x_p(t), u_{p1}(t), \frac{p_{\text{ref}}}{p} u_{p2}(t)\big) \le 0 \\ \not\Longrightarrow\ & h_2\big(x_p(t), u_{p1}(t), u_{p2}(t)\big) \le 0. \end{aligned} \tag{22}$$

Definition 9. (Warpable Reference (WR)) Regard a general warpable system with $p \in [p_{\min}, p_{\max}]$. Consider as well general inequality constraints

$$h_2\big(x(t), u_1(t), u_2(t)\big) \le 0, \quad t \in [0, T], \tag{23}$$

that any feasible trajectory should satisfy. Then, we define the trajectory $y_{\text{wr}}(t)$, obtained for a parameter value p_{wr}, to be a *warpable reference (WR)* if:

(i) $y_{\text{wr}}(t)$ satisfies (23).
(ii) Any warped trajectory of $y_{\text{wr}}(t)$, with warping factor $\dot{w} = p/p_{\text{wr}}$, satisfies (23).

Definition 10. (Best Warpable Reference (BWR)) Regard a general SWOCP as defined by (21) and with $p \in [p_{\min}, p_{\max}]$. A trajectory $y_{\text{bwr}}(\tau)$, obtained for a parameter value p_{bwr}, is defined to be a *best warpable reference (BWR)* if:

(i) $y_{\text{bwr}}(t)$ is an optimal solution of the SWOCP(p_{bwr}).
(ii) $y_{\text{bwr}}(t)$ is a WR with respect to the constraint (23).

Corollary 11. (Optimal Reference for SWOCP). Regard a SWOCP for which a BWR exists and the constraint (21d) is inactive at this BWR. Then, the BWR could be regarded as an optimal reference, i.e. $p_{\text{ref}} = p_{\text{bwr}}$, and any warped trajectory $y_p(t)$, obtained by warping the optimal solution $y^*_{\text{ref}}(\tau)$ of the reference SWOCP(p_{ref})=SWOCP(p_{bwr}), is also an optimal solution of the general SWOCP(p).

Proof. In an optimization problem, any inactive inequality constraint at the optimal solution can be removed from the problem without modifying the local optimal solution (global in case of convex problems). In our case, the u_2-dependent constraint (21d) is inactive at $y^*_{\text{ref}}(\tau)$ and by Definition 9 and 10 any warped trajectory $y_p(t)$ also satisfies (21d). As a result, (21d) can be removed, the original SWOCP is transformed into a WOCP and Corollary 11 holds directly due to Theorem 7.

Remark 12. If h_2 is active for p_{bwr}, i.e. $h_2\big(y_{\text{bwr}}(\tau)\big) = 0$, the warped trajectories $y_p(t)$ are usually suboptimal. In this case, since they are still feasible and are generated from an optimal trajectory, they still represent a better solution than a random feasible trajectory.

Theorem 13. (Existence and Generation of BWRs) Regard the optimal solution of the SWOCP($p_{\max}$) to be $y^*_{\max}(\tau)$. Regard as well m inequality constraints involving u_2, i.e. $h_2(x, u_1, u_2) = [h_{2,1}(\cdot), h_{2,2}(\cdot), \ldots, h_{2,m}(\cdot)]$. If $h_{2,i}\big(x(\tau), u_1(\tau), u_2(\tau)\big)$, $\forall\ i = 1, \ldots, m$ and $\forall\ \tau \in [0, \bar{\tau}]$, is monotonically increasing (decreasing) with respect to u_2 and is only active for $u_2 \ge 0$ ($u_2 \le 0$), then $y^*_{\max}(\tau)$ is a BWR.

Proof. Since $y^*_{p_{\max}}(\tau)$ is an optimal solution, it satisfies the constraint $h_{2,i}(\cdot) \le 0$. Furthermore, using the standard warping relations (14a-14c), feasibility is equivalent to saying that any warped trajectory $y_p(t)$ satisfies:

$$h_{2,i}\big(x_p(t), u_{p1}(t), \frac{p_{\max}}{p} u_{p2}(t)\big) \le 0. \tag{24}$$

Moreover, for any monotonically increasing (decreasing) $h_{2,i}$ and positive (negative) values of u_2 it holds that:

$$h_{2,i}\big(x_p(t), u_{p1}(t), u_{p2}(t)\big) \leq h_{2,i}\big(x_p(t), u_{p1}(t), \frac{p_{\max}}{p} u_{p2}(t)\big) \tag{25}$$

Finally, combining (24–25) and using the fact that h_2 is only active for positive (negative) u_2 values, it holds that:

$$h_{2,i}\big(x_p(t), u_{p1}(t), u_{p2}(t)\big) \leq 0. \tag{26}$$

As a result, any warped trajectory $y_p(t)$ is a feasible solution with respect to $h_{2,i}(\cdot)$, $\forall\ i = 1, \ldots, m$, and $y^*_{p_{\max}}(\tau)$ is a BWR.

Remark 14. An important class of functions satisfying the above equations is any $h_2\big(x(t), u_1(t), u_2(t)\big)$ that can be reformulated as:

$$\begin{aligned} &f_{\min}\big(x(t), u_1(t)\big) \leq u_2(t) \leq f_{\max}\big(x(t), u_1(t)\big), \\ &\text{where}: \quad f_{\min}(x, u_1) \leq 0, \quad f_{\max}(x, u_1) \geq 0. \end{aligned} \tag{27}$$

It should be finally mentioned, that there are SWOCPs, for which no BWR exists. The applicability of warping to those kind of SWOCPs and generation of reasonable WRs go far beyond the scope of this paper.

4. WARPING NMPC

Considering a warpable system in the field of tracking NMPC, Theorems 4 and 7 have an implication on the controller stability and efficiency. In particular for online generation of feasible and optimal tracking trajectories as a function of $p(t)$, we introduce a novel algorithm called *Warping NMPC* for the specific set of warpable systems.

4.1 Generation of Feasible Trajectories

A first variant of warping NMPC exploits Lemma 4 for warpable dynamical systems. In particular, making use of Definition 9, any WR of the system can be used as reference $y_{\text{ref}}(\tau) = y_{\text{wr}}(\tau)$ and then time warped to obtain feasible tracking trajectories $y_p(t)$ for any $p(t)$ value.

Figure 1 illustrates the warping NMPC concept: the reference trajectory y_{ref}, feasible for a constant p_{ref}, is computed offline. Then, by time warping y_{ref} online, warping NMPC generates a feasible tracking trajectory for the real $p(t)$. In a computer implementation, a discrete precomputed reference trajectory $Y_{\text{ref}} = (Y_{\text{ref},0}, \ldots, Y_{\text{ref},N})$ is used to obtain the discrete tracking trajectory at the current time $Y_{\text{track}} = (Y_{\text{track},0}, \ldots, Y_{\text{track},N})$.

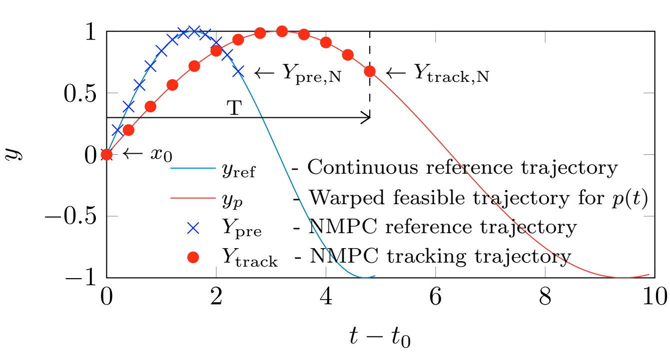

Fig. 1. Warping NMPC example with $\dot{w}(t) = \frac{p(t)}{p_{\text{ref}}} = \frac{1}{2}$.

4.2 Generation of Optimal Trajectories

Feasibility is in many cases a requirement that is strong enough to ensure stability of the tracking NMPC scheme. However, if optimality is desired, the following two variants of warping NMPC can be regarded:

(i) For a WOCP(p), any warping-generated tracking trajectory $y_p(t)$ from an optimal reference trajectory $y^*_{\text{ref}}(\tau)$ will also be optimal due to Theorem 7.

(ii) In case of a SWOCP(p), an existing BWR can be used as reference trajectory $y^*_{\text{ref}}(\tau)$ with $p_{\text{ref}} = p_{\text{bwr}}$. Then, by Corollary 11, the warped tracking trajectories $y_p(t)$ will be either optimal or suboptimal depending on whether $h_2(\cdot)$ is inactive or active at $y^*_{\text{ref}}(\tau)$. In the latter case, since the trajectories are warped from an optimal trajectory, they still represent a better solution than a random feasible trajectory. However, before a real implementation, a concrete evaluation of their quality should be performed.

4.3 Implementation

For a real implementation, the reference trajectory $y^*_{\text{ref}}(\tau)$ is defined as a discrete reference trajectory Y_{ref} on a reference time grid $\tau_{\text{ref}} = [\tau_{\text{ref},0}, \tau_{\text{ref},1}, \ldots, \tau_{\text{ref},M}]$. The warped tracking trajectory is denoted as Y_{track}, the controller sampling time as Δt and the current time point in the warped time frame as τ_0. Considering a NMPC horizon of $N+1$ points, the algorithm to generate optimal and feasible tracking trajectories is given by Algorithm 1.

Algorithm 1 Warping NMPC

1: **Global** $\tau_0 = 0$
2: **function** NEWTRACKINGTRAJECTORY(p)
3:     $Y_{\text{track}} \leftarrow [\ ]$
4:     **for** $i = 0 : N$ **do**
5:         $\tau_i \leftarrow \text{mod}(\tau_0 + i\,\Delta t\,\frac{p}{p_{\text{ref}}}, \tau_{\text{ref},M})$
6:         $Y_{\text{track}} \leftarrow [Y_{\text{track}}, \text{NEXTPOINT}(\tau_i, p)]$
7:     **end for**
8:     $\tau_0 \leftarrow \tau_0 + \Delta t\,\frac{p}{p_{\text{ref}}}$
9:     **return** Y_{track}
10: **end function**
11: **function** NEXTPOINT(τ_{next}, p)
12:     $\tau_{\text{low}} \leftarrow \arg\min_{\tau} |\tau - \tau_{\text{next}}|, \quad \text{s.t. } \tau \leq \tau_{\text{next}},\ \tau \in \tau_{\text{ref}}$
13:     $\tau_{\text{up}} \leftarrow \arg\min_{\tau} |\tau - \tau_{\text{next}}|, \quad \text{s.t. } \tau > \tau_{\text{next}},\ t \in \tau_{\text{ref}}$
14:     $y_{\text{low}} \leftarrow Y_{\text{ref}}(\tau_{\text{low}}), \quad y_{\text{up}} \leftarrow Y_{\text{ref}}(\tau_{\text{up}})$
15:     $y_{\text{next}} \leftarrow y_{\text{low}} + \frac{y_{\text{up}} - y_{\text{low}}}{\tau_{\text{up}} - \tau_{\text{low}}}(\tau_{\text{next}} - \tau_{\text{low}})$
16:     $y_{\text{next}}(u_2) \leftarrow y_{\text{next}}(u_2)\,\frac{p}{p_{\text{ref}}}$
17:     **return** y_{next}
18: **end function**

The main principle of the algorithm is to track the current time point τ_0 in the warped time frame and iteratively build the tracking trajectory starting from τ_0. In particular, the time location of the i^{th} trajectory point is first computed at τ_i in the reference time grid τ_{ref} by using a warped incremental time step $\Delta t\, p/p_{\text{ref}}$. Then, the i^{th} tracking point is computed by interpolation between the closest points of τ_i in the reference grid τ_{ref}. In addition, the u_2-values of y_{new} are amplified or attenuated accordingly. It is important to remark that, without loss of generality, we have used a periodic reference trajectory (line 5).

5. APPLICATION EXAMPLE

In this section, warping NMPC is illustrated and tested on an *airborne wind energy (AWE)* system based on a tethered kite flying periodic optimal trajectories and generating energy (Fagiano et al. (2009)). The idea is to use NMPC for tracking optimal trajectories that maximize the extracted energy in a so-called pumping cycle. More details on power generation of AWE can be found in Ahrens et al. (2013).

5.1 Kite Dynamics as Warpable System

The kite model is based on four states as illustrated in Fig. 2: the tether length l in combination with two polar coordinates ϑ and φ for the kite position, and then, an angle ψ for the orientation.

Following Erhard and Strauch (2013, 2015), where details on the model can be found, the subsequent equations of motion can be derived:

$$\dot{\psi} = v_{\mathrm{w}} \cos\vartheta\, E\Big(g_{\mathrm{k}}\,\delta - \frac{\cot\vartheta \sin\psi}{l}\Big) + v_{\mathrm{reel}}\, E\Big(\frac{\cot\vartheta}{l} - g_{\mathrm{k}}\,\delta\Big) \quad (28a)$$

$$\dot{\varphi} = v_{\mathrm{w}} \frac{-E\cot\vartheta\sin\psi}{l} + v_{\mathrm{reel}} \frac{E}{l\sin\vartheta}, \quad (28b)$$

$$\dot{\vartheta} = v_{\mathrm{w}} \frac{-\sin\vartheta + E\cos\vartheta\cos\psi}{l} - v_{\mathrm{reel}} \frac{E\cos\psi}{l}, \quad (28c)$$

$$\dot{l} = v_{\mathrm{reel}}. \quad (28d)$$

The kite steering input δ and the tether reeling speed v_{reel} form the system control input vector $[\delta, v_{\mathrm{reel}}]^\top$, the glide ratio E and steering proportionality constant g_{k} are system parameters, and v_{w} is the ambient wind velocity.

Analyzing the structure of (28a–28d), it can be recognized that the kite is a warpable system, where $u_1 = \delta$, $u_2 = v_{\mathrm{reel}}$ and $p = v_{\mathrm{w}}$. In particular, the wind velocity v_{w} as warping parameter determines the speed of the system dynamics. As a result, online generation of feasible and optimal trajectories for different v_{w} values can be done with the previously described warping NMPC algorithm in order to implement a stable and robust tracking NMPC scheme.

5.2 Optimal Trajectories

As defined by Erhard et al. (2017), the optimal periodic trajectories $y^*(t) = (x^*(t), u^*(t))$, maximizing the average power in a pumping cycle for a given wind velocity value v_{w}, are obtained by:

$$\underset{y(\cdot)}{\text{minimize}} \quad -\frac{1}{T}\int_0^T v_{\mathrm{a}}(t)^2\, \dot{l}(t)\, \mathrm{d}t \quad (29a)$$

$$\text{subject to} \quad \Phi(x(t), u(t), v_{\mathrm{w}}) = \dot{x}(t), \quad t \in [0, T], \quad (29b)$$

$$h\big(x(t), \delta(t)\big) \le 0, \quad t \in [0, T], \quad (29c)$$

$$v_{\min} \le v_{\mathrm{reel}}(t) \le v_{\max}, \quad t \in [0, T], \quad (29d)$$

$$r\big(x(T), x(0)\big) \le 0. \quad (29e)$$

with $\Phi(\cdot)$ given by (28a–28d), $v_{\min} \le 0$, $v_{\max} \ge 0$ and $v_{\mathrm{a}} = v_{\mathrm{w}}\, E\, \cos\vartheta - \dot{l}\, E$. By expanding the cost function as:

$$J = -\frac{1}{T}\int_0^T v_{\mathrm{a}}^2\, \dot{l}\, \mathrm{d}t = -\frac{1}{T}\int_0^T \big(v_{\mathrm{w}}\, E\cos\vartheta - v_{\mathrm{reel}}\, E\big)^2 v_{\mathrm{reel}}\, \mathrm{d}t$$

$$= -\frac{v_{\mathrm{w}}^3}{T}\int_0^T \left(E\cos\vartheta - \left(\frac{v_{\mathrm{reel}}}{v_{\mathrm{w}}}\right)\right)^2 \left(\frac{v_{\mathrm{reel}}}{v_{\mathrm{w}}}\right) \mathrm{d}t \quad (30)$$

it can be observed that the OCP has the same structure as (21) with $u_2(t) = v_{\mathrm{reel}}(t)$ and $p = v_{\mathrm{w}}$, and thus, the optimal tracking trajectories are the solution of a SWOCP.

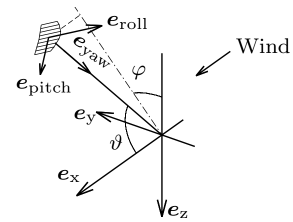

Fig. 2. Kite coordinate system (Erhard et al. (2017)).

As a side result, the scaling of AWE power output with the cube of wind velocity, a result known for conventional wind turbines, can be directly concluded from (30).

5.3 Warping Illustration

To have a graphical representation of the warping properties of the kite system, Fig. 3 depicts the 3D view of the optimal trajectories for three different v_{w} values. Figure 4 illustrates the same solutions in time domain for two system states. We can observe how, in a 3D space, the three trajectories make the kite fly through the same physical locations, but, in the time domain, the kite dynamics have different velocities and the state trajectories are warped versions of each other.

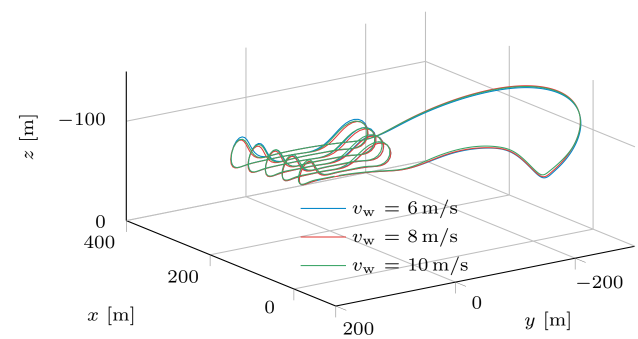

Fig. 3. Optimal solutions y^* for different v_{w} values.

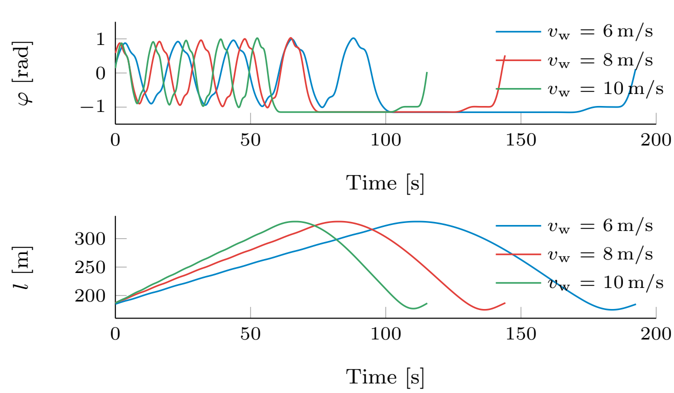

Fig. 4. OCP system states φ^* and l^* for different v_{w} values.

To further comprehend the warping properties, Fig. 5 represents the optimal control inputs of the previous trajectories. As expected, since v_{reel} represents a u_2-type input, it does not just warp but is also attenuated or amplified. By contrast, the control δ is just warped.

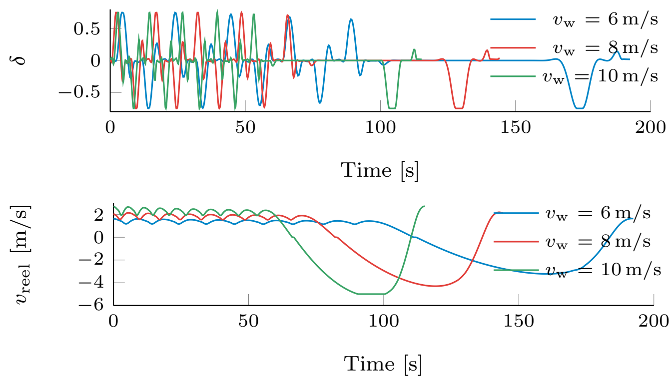

Fig. 5. OCP system controls for different v_w values.

5.4 Optimality of Warped Solution

Since the tracking trajectories are optimized using a SWOCP and since (29d) has the structure of (27), by Theorem 13 we know that a BWR can be computed using the maximum v_w-value (15 m/s in this case) as a reference.

Since at the optimal solution y^*_{ref} for 15 m/s the constraint (29d) is active, the warped tracking trajectories y_p are suboptimal (refer to Remark 12). Therefore, to evaluate the decrease in optimality of y_p, Table 1 compares the power efficiency of optimal trajectories at different $p = v_w$ values with respect to their warped counterparts.

Table 1. Efficiency comparison of optimal solutions and trajectories obtained by warping.

v_w	6 m/s	8 m/s	10 m/s	12 m/s	14 m/s	15 m/s
η_{Loyd} optimal	35.4 %	35.4 %	35.3 %	34.9 %	34.2 %	33.7 %
η_{Loyd} warping	33.7 %					

The power efficiency η_{Loyd} is computed as the ratio between the extracted average power divided by the maximum ideal power as defined by Loyd (1980):

$$\eta_{Loyd} = \frac{J}{\frac{4}{27} E^2 v_w^3} \tag{31}$$

with J given by (30) and the nominator adjusted accordingly to the model (Erhard et al. (2017)).

Considering that explicitly solving the SWOCP for different v_w values leads only to a maximum efficiency increase of less than 2%, the warped trajectories represent a very good approximation of their optimal counterparts. Therefore, warping NMPC is a highly efficient algorithm for online generation of nearly optimal trajectories for this AWE system.

5.5 Warping NMPC

In order to assess the tracking performance of warping NMPC, the controller is tested against a wind speed profile decreasing in time from 10 m/s to 6 m/s. Figure 6 depicts the wind profile as well as the 3D pumping cycle trajectories at the end of the simulation interval. It can be observed that normal NMPC, which is based on a constant tracking trajectory generated at 10 m/s, is unable to track the reference trajectory and extract energy (indicated by a negative Loyd factor $\eta_{Loyd} = -2.09\%$). In particular, it keeps the kite at a high elevation angle and barely performs any movement. By contrast, the warping NMPC reaches power efficiencies ($\eta_{Loyd} = 31.57\%$) very close to the ideal one by adaptation to the varying wind speed v_w.

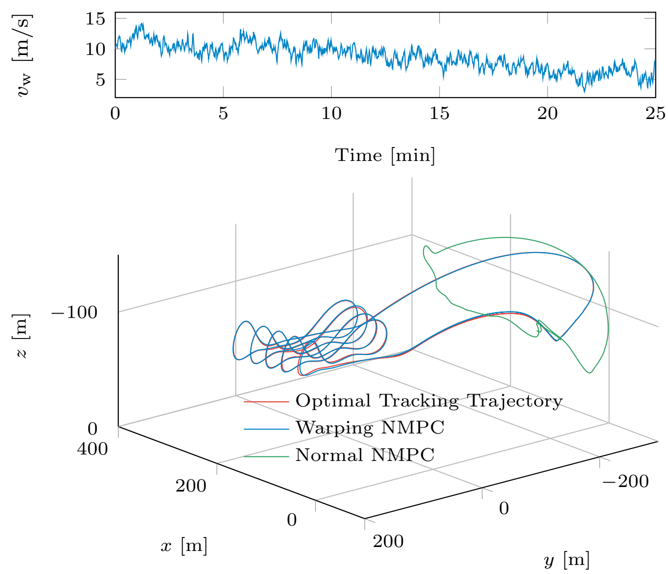

Fig. 6. Comparison between normal NMPC and warping NMPC for $v_w = 6$ m/s.

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